Performance Evaluation, Contracts, and Flows in Efficient Markets

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Abstract

A common misconception in finance is that investors who believe in an efficient market will not invest with an active fund manager. In this paper I develop a model of active money management in which both the investor and the fund manager believe that the market is efficient and that all assets are fairly priced. Furthermore traditional performance measures would not be positive for the manager. But despite this the investor still finds it desirable to invest with the manager. The model yields equilibrium contracts, performance measures, and performance/flow relationships that conform with those we observe in the mutual fund industry.

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G11, G14

1. Introduction

A common misconception in finance, even among academics, regards the relationship between a belief in an efficient market and a willingness to hire
an active manager. The common view seems to be that if the market is efficient then active managers cannot generate performance which justifies their higher fees, compared to index funds. Those who espouse this view express surprise that such a high percentage of investor wealth is invested with active managers.¹

The purpose of this paper is to demonstrate that this view is incorrect. I construct a model in which market participants, both investors and managers, observe the same information and agree that all assets are fairly priced and that the market is efficient. But investors are still willing to pay active managers for their services. Importantly I will not assume that the investor can only invest through the manager. In fact both investor and manager have free access to trading in all the available assets.

The lack of empirical evidence for superior performance of managed funds is interpreted as a reason not to invest in such funds. The problem with this view is that it relies on a particular definition of performance which, I will show, does not necessarily correspond with what investors would use when choosing their investments.

The approach to performance evaluation taken in the academic literature involves using some asset pricing model to computed risk-adjusted excess returns. If the model correctly prices all the primitive assets, in the sense that their risk-adjusted excess returns are all zero, then the risk-adjusted excess return of any portfolio of primitive assets should also be zero, unless the portfolio strategy involves information superior to that which the model assumes market participants have access to (see the recent review in Ferson

¹See, for example the discussion in Malkiel (2004).
(2012) for details of this argument). So what academics call performance measurement is really an attempt to identify which managers possess superior information.

The model in this paper demonstrates that it is not necessary for a manager to possess, or even claim to possess, such superior information in order for investors to find it advantageous to hire that manager. In the model both the manager and investor have access to the same information. The key insight is that even in a world in which equally informed traders agree that all assets are fairly priced and that the market is efficient they may still disagree about the distribution of asset returns. If so then they will disagree about the best way to combine these assets into a portfolio.

The reason for disagreement between agents who are equally informed has to do with what Kim and Verrecchia (1994) call information processing skill. Information processing occurs when an agent converts public information into a private forecast. In my model investors are only partially convinced that any given manager is skilled at information processing and so are willing to allocate only a part of their wealth to the manager’s portfolio.

In a dynamic setting investors will update about the skill of managers as returns are realized. As their beliefs change they will change their allocation to the actively managed portfolio accordingly. But even if they become more convinced that the manager is skilled they still perceive the managed portfolio as not providing superior risk-adjusted returns. In this way we can resolve the puzzle first noted by Gruber (1996) that while empiricists have not found strong evidence for consistent, superior performance of active managers, investors still seem to “chase performance” by shifting money to
funds that have recently beat their benchmarks. In my model this is perfectly rational and such excess returns become the natural performance measures. In fact the model yields insights into why practitioners focus on excess returns over (appropriately chosen) benchmark portfolios rather than using an asset pricing model to risk-adjust observed returns as academics have favored.

To be clear, I am not arguing in this paper that managers can or cannot achieve superior risk-adjusted returns. I will simply show that in a world in which there is no superior risk-adjusted performance, updating about manager skill, as defined above, can account for some of the phenomena and practices that we observe in the mutual fund industry. In particular I will show that the nature of the equilibrium contracts we observe conform to industry practice and that the performance/flow relationship which emerges is also consistent with what has been found empirically.

Because in this model the market for portfolio management services is characterized by asymmetric beliefs about manager skill we must address the issue of why, in equilibrium, this asymmetry in beliefs can persist. If the issue was simply one of hidden type, with some unskilled managers pretending to be skilled, then we would expect a separating equilibrium in which types are revealed by the contract choice of the players. Such an equilibrium has been examined by Bhattacharya and Pfleiderer (1985) and Heinkel and Stoughton (1994) but the contracts which arise do not resemble those we observe. So such screening on type doesn’t seem to be what happens in the market for portfolio managers.

Now suppose that investors believe that only a subset of managers is skilled but that each manager believes with probability 1 that he or she is
in that subset. This “dogmatic” prior might be seen as an extreme form of overconfidence on the part of managers. A given manager might very well be incompetent but there is no empirical evidence that could convince the manager of this. Screening would be a futile exercise because while every manager would report their type as “skilled” the investor would not learn anything by this report.

Accordingly this equilibrium must involve updating on the part of investors about the skill of managers. Equilibria in which agents have heterogeneous beliefs and update through time have been examined in the literature. The approach adopted in this article is similar to Adrian and Westerfield (2009) which develops a dynamic contracting model between parties with different beliefs, although in a different context.

Berk and Green (2004) also consider a world in which investors update about what performance managers are capable of producing. In that analysis neither the financial market nor the portfolio choice of the manager is modeled. Investors’ behavior is also not modeled explicitly except that flows are assumed to result from updating. The size of the fund is determined by diseconomies of scale which drive down expected returns when the fund grows. There are a few errors in the Berk and Green (2004) analysis which I will not cover in detail.\(^2\)

\(^2\)A full critique is available upon request. Briefly the problems arise because the tradeoff between updating and diseconomies of scale means that size fluctuates with excess returns. The implications of this for flows is different from what Berk and Green (2004) suggest. There is also a problem with their definition of net flows in equation (33) which affects the calibration portion of the paper.
2. The Model

2.1. The Market

In the market there are many investors and managers. It seems reasonable to assume that all investors have the same beliefs. For simplicity I will further assume that all investors also have the same utility. This means that in the model there is only a need for one representative investor. For reasons that will become clear later it will be interesting to examine cases in which there are multiple managers with different beliefs. Let $M$ be the number of managers. All agents maximize a time separable utility of consumption over an infinite horizon. The investor’s consumption at time $t$ will be denoted $c(t)$ while the consumption for the managers, which we think of as coming from their fees, are $\phi_j(t)$ for $j = 1, \ldots, M$.

I will model the financial market as taking place in continuous-time and take full advantage of the tractability this affords. There are $N$ non-redundant, risky assets traded in the market and a locally risk-free asset with return $r(t)$. Agents do not disagree about the (almost surely non-singular) volatility matrix, $\sigma(t)$. However they do disagree about expected returns. The expected return vector corresponding to investor beliefs will be denoted $\mu_I(t)$ and similarly $\mu_j(t)$ will denote the expected return vector of the $j$th manager. The processes $\sigma(t), r(t), \mu_I(t), \mu_j(t)$ are adapted to $\mathcal{F}(t)$, a (right-continuous) filtration of the underlying probability space. All market participants observe the returns on the underlying assets. Since they observe the same returns on all assets but disagree about the expected returns they must also disagree about the innovations in asset returns. The innovation processes are $N$-dimensional standard Brownian motions which we will
denote by $Z_I(t)$ and $Z_j(t)$ for $j = 1, \ldots, M$. The following are equivalent descriptions of the vector of instantaneous returns of the $N$ risky assets.

$$dR(t) = \mu_I(t)dt + \sigma(t) dZ_I(t)$$

$$dR(t) = \mu_j(t)dt + \sigma(t) dZ_j(t) \quad \text{for } j = 1, \ldots, M$$

Define the vector of risk prices perceived by the various agents as

$$\theta_I(t) = \sigma(t)^{-1}(\mu_I(t) - r(t))1, \quad \text{and} \quad \theta_j(t) = \sigma(t)^{-1}(\mu_j(t) - r(t))1 \quad \text{for } j = 1, \ldots, M$$

and we can rewrite the return dynamics as

$$dR(t) = r(t)1dt + \sigma(t)[\theta_I(t)dt + dZ_I(t)]$$

$$dR(t) = r(t)1dt + \sigma(t)[\theta_j(t)dt + dZ_j(t)] \quad \text{for } j = 1, \ldots, M.$$ 

Equating the expressions in square brackets gives us the familiar filtering equation

$$dZ_j(t) = (\theta_I(t) - \theta_j(t)) dt + dZ_I(t) \quad \text{for } j = 1, \ldots, M \quad (1)$$

which relates the innovation processes perceived by the various market participants. This relation is important in the proofs in the appendix.

2.2. Market Efficiency and Performance

As pointed out in Fama (1970) and Fama (1991), statements about market efficiency must be accompanied by a specification of an information set and an equilibrium asset pricing model. The filtration $\mathcal{F}(t)$ is the relevant information set which all market participants observe.

Any asset pricing model can be described by a stochastic discount factor process. From the previous subsection we can see that corresponding to the
market price of risk processes and the innovation processes for the various agents there are actually several stochastic discount factor processes in this model.

\[
\frac{dH_I(t)}{H_I(t)} = -r(t)dt - \theta_I(t)^	op dZ_I(t) \text{ and }
\]
\[
\frac{dH_j(t)}{H_j(t)} = -r(t)dt - \theta_j(t)^	op dZ_j(t) \quad \text{for } j = 1, \ldots, M
\]

In effect the various agents are working with different asset pricing models. But notice that by construction the risk-adjusted excess returns of all primitive assets are zero. This means that all agents perceive that the market is efficient.

At this point it makes sense to ask what performance measure each agent would assign to some of the portfolios which will be of interest in what follows. These portfolios are each maximum Sharpe ratio portfolios as perceived by the various agents. Their dynamics are given by

\[
dR_I(t) = r(t)dt + \theta_I(t)^	op [\theta_I(t)dt + dZ_I(t)]
\]

and

\[
dR_j(t) = r(t)dt + \theta_j(t)^	op [\theta_I(t)dt + dZ_I(t)] \text{ for } j = 1, \ldots, M
\]

An important special case of investor beliefs is when the investor has no confidence in any manager. The beliefs corresponding to this case can be summarized in a risk price vector process \(\theta_B(t)\) and an innovation process \(Z_B(t)\). The return on the maximum Sharpe ratio portfolio corresponding to these beliefs will be denoted \(dR_B(t)\). We will call this portfolio a benchmark portfolio and assume that all participants know it’s composition and observe it’s return each period.
It is important to note that none of the above portfolios exhibit superior risk adjusted performance from the perspective of investors or any of the managers. The return dynamics above are expressed using the investor’s risk prices and innovation processes. From the investor’s perspective these portfolios all have expected returns commensurate with the risk exposures they are taking on. If we switch to the perspective of the managers, by using (1) to rewrite these return dynamics in terms of $\theta_j(t)$ and $dZ_j(t)$ then the same would be true. Similarly any other market observer, say an econometrician testing for investment performance, would find the same, provided that the observer used risk prices and perceived innovations that are consistent with all primitive assets being correctly priced.

3. Equilibrium Fees

I will assume there is no moral hazard in the market so a pareto optimal equilibrium can be found by solving the social planner’s problem with some weights$^3$. Let $\lambda_I$ be the weight for the investor and $\lambda_j, j = 1, \ldots, M$ be the weights for the managers. The social planner’s problem is to choose consumptions for all agents in order to maximize a weighted sum of the expected utilities of the agents subject to the constraint that the present value of the sum of all these consumption streams is equal to aggregate wealth. All wealth is assumed to belong to investors.

If we assume that all agents have time separable expected utilities with a common rate of time preference, $\rho$ then it is shown in the appendix that

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$^3$Equivalently this equilibrium could be obtained by maximizing one player’s utility subject to participation by all the other players.
the first-order conditions of the problem are

\[ u'_I(c(t)) = e^{\rho t} \frac{y}{\lambda_I} H_I(t) \]

\[ u'_j(\phi_j(t)) = e^{\rho t} \frac{y}{\lambda_j} H_j(t) \]

where \( y \) is the multiplier on the budget constraint. Notice that these first-order conditions do not satisfy the so-called Borch rule in that the ratio of marginal utilities is not constant. Instead we have that the ratio of marginal utilities is given by the martingale

\[ d \left( \frac{H_I(t)}{H_M(t)} \right) = \frac{H_I(t)}{H_M(t)} (\theta_M(t) - \theta_I(t))^\top dZ_I(t). \] (2)

This martingale will be important in what follows.

Given a form for the marginal utilities we can solve for the equilibrium investor consumption and fee for each manager. We wish to interpret these solutions in terms of the quantities which we observe, i.e. portfolio returns, assets under management, and flows into and out of the managed funds. In order to do so I consider several special cases.

3.1. Equilibrium contracts under log utility

If all agents have log utility then equilibrium consumption and fees are

\[ c(t) = \frac{\lambda_I e^{-\rho t}}{yH_I(t)} \text{ and } \phi_j(t) = \frac{\lambda_j e^{-\rho t}}{yH_j(t)}. \] (3)

Applying the Ito formula we have that

\[ \frac{dc(t)}{c(t)} = -\rho dt + dR_I(t) \]

and

\[ \frac{d\phi_j(t)}{\phi_j(t)} = -\rho dt + dR_j(t). \]
Computing the value of the consumption stream and the fees and adding we have that total wealth is given by

\[ W(t) = \frac{c(t)}{\rho} + \sum_{j=1}^{M} \frac{\phi_j(t)}{\rho} \]

with dynamics

\[ dW(t) = -\left( c(t) + \sum_{j} \phi_j(t) \right) dt + \frac{c(t)}{\rho} dR_I(t) + \sum_{j} \frac{\phi_j(t)}{\rho} dR_j(t). \]

For the moment imagine that there is only one manager. The investor considers that there are only two possible scenarios: (1) the manager is skilled and so the strategy corresponding to \( \theta_M(t) \) gives the best possible portfolio or (2) the manager is not skilled and so \( \theta_B(t) \) gives a better portfolio. At time 0 the investor believes that the manager is skilled with probability \( p(0) \). But over time the investor observes returns in the market and updates this probability and so at time \( t \) the investor’s beliefs about risk prices are \( \theta_I(t) = p(t)\theta_M(t) + (1 - p(t))\theta_B(t) \). From this we have that

\[ dR_I(t) = p(t)dR_M(t) + (1 - p(t))dR_B(t) \]

which says that the investor’s optimal portfolio is a position in both the managed portfolio and the benchmark with weights given by the respective probabilities that each portfolio is the true maximum Sharpe ratio portfolio.

Substituting this expression for \( dR_I(t) \) into the wealth dynamics we obtain

\[ dW(t) = -(c(t) + \phi(t))dt + \frac{c(t)p(t) + \phi(t)}{\rho} dR_M(t) + \frac{c(t)(1 - p(t))}{\rho} dR_B(t). \]

The expression multiplying \( dR_M(t) \) is the amount of wealth invested in the strategy that the manager believes is best. In industry parlance this is called “assets under management”. I will denote this by \( A(t) \).
It seems natural to refer to $p(t)$ as the manager’s reputation. It is shown in the appendix that the updating rule for $p(t)$ is

$$p(t) = p(0) \frac{H_I(t)}{H_M(t)}. \quad (4)$$

Using (3) and (4) we see that the fee received by the manager is a fixed fraction of assets under management.

$$\frac{\phi(t)}{A(t)} = \frac{\rho}{\phi(t)p(t) + 1} = \frac{\rho}{\lambda p(0) + 1}$$

The intuition for this result has to do with optimal risk sharing. From the investor’s perspective the manager’s reputation is a martingale. But from the manager’s perspective this is not the case, the manager believes that the data will eventually convince the investor of the manager’s skill. So the cheapest way for the investor to compensate the manager is to let the manager bet on his or her own reputation by agreeing to a pay schedule which fluctuates with reputation. This was also noted by Adrian and Westerfield (2009) who showed that an optimal contract between parties with differing beliefs will involve a “side bet” involving the relative likelihoods that certain events will occur.

This has an appealing similarity to the contracts that we actually observe. In practice fund managers do not change their fees, relying instead on the growth of assets under management to reap the reward, as they suppose, of their skill.\footnote{Actually this is a bit of an overstatement. Warner and Wu (2011) document that when mutual fund advisory contracts are renewed by the fund board there is occasionally a small change of fee.}
3.2. Updating and performance measures

I will employ a mild abuse of notation and express portfolio returns as

\[ dR_j(t) = \mathbb{E}_t^I [dR_j(t)] + \theta_j(t)^\top dZ_I(t), \quad \text{for } j = M, B \]

where the subscript on the expectation operator indicates conditioning on \( \mathcal{F}(t) \). Using these expressions allows us to rewrite the dynamics of \( p(t) \) (given in the appendix) as

\[ dp(t) = p(t)(1 - p(t)) \left( dR_M(t) - dR_B(t) - \mathbb{E}_t^I [dR_M(t) - dR_B(t)] \right). \] (5)

The above says that updating depends on the excess return of the managed portfolio over the benchmark return, adjusted for differences in expected return. This may explain why practitioners are interested in excess returns over a benchmark as a performance measure rather than the performance measures favored in academia.

In Berk and Green (2004) updating also depends on excess returns over a benchmark but that was an assumption of their model. Here it is a result.

Notice that a given excess return does not always lead to the same increase in reputation. The maximum responsiveness of reputation to performance happens when \( p(t) = 0.5 \). At higher or lower levels of \( p(t) \) changes in reputation are harder to come by.

3.3. Competition among managers

Generalizing the model to allow for multiple managers is straightforward. To make the problem interesting the managers must have different beliefs about risk prices. To begin with suppose that each manager believes that he or she is the only skilled manager and that the investor believes that at most
one of the managers could be skilled. So the investor’s inference problem is to guess which (if any) of the managers has skill.

Let $p_j(t)$ be the probability assessed by the investor at time $t$ of whether manager $j$ is skilled. Provided all agents have log utility we have that total wealth dynamics are

$$dW(t) = -\left(c(t) + \sum_{j=1}^{M} \phi_j(t)\right) dt + \sum_{j=1}^{M} \frac{c(t)p_j(t) + \phi_j(t)}{\rho} dR_j(t)$$

$$+ \frac{c(t)\left(1 - \sum_{j=1}^{M} p_j(t)\right)}{\rho} dR_B(t)$$

such that assets under management for the $j$th manager is

$$A_j(t) = \frac{c(t)p_j(t) + \phi_j(t)}{\rho}.$$ 

As in the one manager case we have that

$$\frac{p_j(t)}{p_j(0)} = \frac{H_1(t)}{H_j(t)}$$

and once again the equilibrium fee paid to each manager is a constant fraction of the assets under their own management.

The updating is somewhat more interesting than in the one manager case because simple excess returns over the benchmark are not sufficient. Instead the excess returns of all other managers matter. The expression for the percentage change in each manager’s reputation is

$$\frac{dp_j(t)}{p_j(t)} = dR_j(t) - dR_I(t) - \mathbb{E}_t \left[ dR_j(t) - dR_I(t) \right] \quad \text{for} \quad j = 1, \ldots, M$$

but since

$$dR_I(t) = \sum_{j=1}^{M} p_j(t) dR_j(t) + \left(1 - \sum_{j=1}^{M} p_j(t)\right) dR_B(t)$$
we have that the relevant measure is the excess return over a weighted average of returns of the benchmark and the manager’s peer group. Whether this corresponds to industry practice is hard to say. Certainly managers are compared to their peers as well as to their benchmarks. Whether such comparisons take the form above is an open question.

4. Flows and Power Utility

Since the manager’s fee is a fraction of assets under management it is important to know how assets under management changes with performance. Applying the Ito formula to the expression for $A(t)$ from the one manager case gives us the following.

$$dA(t) = p(t)\frac{dc(t)}{\rho} + \frac{d\phi(t)}{\rho} - d\rho(t) + \frac{1}{\rho} \langle dc(t), dp(t) \rangle$$

Using the expressions derived above for the dynamics of consumption, fee, and reputation we obtain

$$\frac{dA(t)}{A(t)} = -\rho dt + dR_M(t).$$

Recall that flows are defined as the percentage changes in assets under management net of investment return. From the above we can see that in this model flows would simply be $-\rho dt$. The only flows are consumption and fee withdrawals. But since these are independent of performance there are no performance related flows in this model. This illustrates an important point that has been missed by reduced-form models such as Berk and Green (2004): updating by itself does not necessarily generate a performance/flow relationship.
Interestingly this is also true in the multiple manager model. There are no flows between competing managers or between any manager and the benchmark. The investor makes an initial allocation among managers and the benchmark depending on $p_j(0)$ and then never reallocates.

The intuition for this is as follows. In the one manager case the initial allocation is between the benchmark and the managed portfolio. If the manager beats the benchmark then the investor updates in the positive direction and now desires a higher allocation to the managed portfolio compared to the benchmark. But since the manager beat the benchmark the allocation has already changed in that direction. For log utility this exactly matches the desired change and so there are no flows. Similar logic works for the multiple manager case.

The fact that we do observe performance driven flows is evidence against the model I have presented above. However it turns out that a slight generalization of the model, to power utility, does result in flows while still retaining many of the desirable features of the log utility case. However it is more difficult to show because closed-form solutions are not available, which is why I started with the log utility case.

In the log utility case there was no need to specify the processes which risk prices and interest rates followed because the model could be solved in general. For power utility we must be more specific about these processes. Suppose for instance that in addition to updating risk there are changes in the investment opportunity set that are driven by some state variable $x(t)$ such that $p(t)$ and $x(t)$ are jointly Markovian. In the one manager case we
would have the following expression for total wealth

\[ W(t) = c(t)G(p(t), x(t)) + \phi(t)G(1, x(t)) \]

where \( G(p, x) \) is the wealth/consumption ratio for a power utility investor which solves a PDE derived in the appendix. Notice that because of the manager’s beliefs this function must be evaluated at \( p = 1 \) to calculate the present value of manager fees. Generalizing to power utility complicates matters a bit because the investor’s portfolio will include an additional component, a hedge portfolio. For utility functions other than log the investor wishes to hedge against changes in the investment opportunity set. The size of this hedging demand is determined by the shape of the function \( G \). The wealth of the investor is now split across the managed portfolio, the risk-free rate, the benchmark portfolio, and a hedge portfolio. But for my purposes the important thing is the amount invested in the managed portfolio, which is given by

\[ A(t) = \frac{c(t)G(p(t), x(t))p(t)}{\gamma} + \frac{\phi(t)G(1, x(t))}{\gamma} \]

where \( \gamma \) is the relative risk aversion of the investor.

To derive the dynamics of \( A(t) \) one must specify the dynamics of \( x(t) \) and how it affects the investment opportunity set so that the PDE for \( G \) can be solved. A convenient specification is to imagine that \( \theta_B \) is constant and that the manager’s risk price vector fluctuates around this vector. Suppose the state variable \( x \) solves the following SDE

\[ dx(t) = x(t)(1 - x(t))\sigma_x^T dZ(t) \]

which restricts \( x \) to values between 0 and 1. Now suppose that at time \( t \) we
have
\[ \theta_M(t) = \theta_B + (2x(t) - 1)D \]
where \( D \) is a constant vector. Notice that for \( x(t) = 1 \) we have \( \theta_M(t) = \theta_B + D \), for \( x = 0.5 \) we have \( \theta_M(t) = \theta_B \), and for \( x = 0 \) we have \( \theta_M(t) = \theta_B - D \).

If we simulate according to this model (assuming a constant risk-free rate) then we can calculate \( A(t) \) for each simulated \( x(t) \) and \( p(t) \) and also calculate \( \phi(t) \) using the first-order conditions for power utility. The result is that \( \phi(t) \) still seems to be a constant fraction of assets under management.\(^5\)

There are performance driven flows in this model. Below I will show simulation results which investigate the nature of the performance/flow relationship. But to build intuition consider applying the Ito formula to the expression for \( A(t) \) above. The simulations suggest that \( G \) is close to flat (implying that hedging demand is small) so to simplify the resulting expression we will ignore any terms that involve derivatives of \( G \). It is shown in the appendix that we obtain the following.

\[
dA(t) = -\frac{c(t)p(t) + \phi(t)}{\gamma}dt + A(t) \left[ \frac{\gamma - 1}{\gamma}r dt + \frac{1}{\gamma}dR_M(t) \right] \\
+ \frac{\gamma - 1}{\gamma}c(t)Gp(t)(1-p(t))(dR_M(t) - dR_B(t)) - \mathbb{E}_t[dR_M(t) - dR_B(t)].
\]

If we let \( \gamma = 1 \) then we obtain the expression for log utility. The term on the second line is the performance driven flows. If \( \gamma > 1 \) then positive excess returns result in flows into the managed fund.

Notice that conditional on reputation the above expression for flows appears to be linear in performance. However this is only true for instantaneous

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\(^5\)Within the simulation I investigate this by regressing the fee fraction on polynomial functions of the state vector. For no specification was the \( R^2 \) even as high as 1%.
changes. Typically empirical researchers use much longer horizons on which to examine flows. For my simulation I take the horizon to be 1-year, the same as Chevalier and Ellison (1997). To compute the benchmark and managed return over this horizon I compute ending portfolio values $V_B(T)$ and $V_M(T)$ according to

$$\frac{dV_B(t)}{V_B(t)} = dR_B(t), \quad \text{and} \quad \frac{dV_M(t)}{V_M(t)} = dR_M(t)$$

up to time $T$ (one year). Consistent with the literature I then define flows as

$$Flows = \frac{A(T) - A(0)}{A(0)} - \frac{V_M(T) - V_M(0)}{V_M(0)}$$

and the excess return over the horizon as

$$Excess = \frac{V_M(T) - V_M(0)}{V_M(0)} - \frac{V_B(T) - V_B(0)}{V_B(0)}.$$

Using this simulated data I then estimate the performance/flow relationship

$$Flows_j = a + b_1Excess_j + b_2Excess_j \times (Excess_j > 0) + e_j$$

where $(Excess_j > 0)$ is a dummy variable which is 1 when the excess return of fund $j$ is positive and zero otherwise. The dummy variable allows me to test the significance of any convexity or concavity in the performance/flow relationship.

I used the following parameter values: $\theta_B = [.25 .25]^T$, $D = [.05 -.05]^T$, $r = .04$, $\rho = .06$, $\gamma = 3$, and $\sigma_x = [-.03 .03]^T$. I take the initial distribution of $x$ to be uniform on $[0, 1]$. I repeat the regression for various starting distributions of manager reputation. Each starting distribution is uniform on the interval indicated at the top of the column. Each regression uses 3000 simulated fund-year observations.
Coefficient | (0,0.2) | (0.2,0.4) | (0.4,0.6) | (0.6,0.8) | (0.8,1.0) \\
---|---|---|---|---|---
\(a\) | -0.0222 | -0.0256 | -0.0288 | -0.0323 | -0.0359 \\
| (0.0001) | (0.0001) | (0.0001) | (0.0001) | (0.0001) \\
\(b_1\) | 1.3499 | 1.0650 | 0.7756 | 0.4845 | 0.1827 \\
| (0.0059) | (0.0062) | (0.0066) | (0.0074) | (0.0076) \\
\(b_2\) | 0.2388 | 0.1311 | 0.0106 | -0.0949 | -0.1212 \\
| (0.0158) | (0.0144) | (0.0150) | (0.0173) | (0.0191) \\

Table 1: Estimates from the regression of simulated flows on excess returns. Standard errors are in parentheses. The coefficient \(b_2\) measures the difference in the response of flows when excess returns are positive compared to when excess returns are negative. A significantly positive \(b_2\) indicates a convex performance/flow relationship.

There are several things to note about the table. As might be expected the coefficient \(b_1\) is always positive and significant so that flows respond positively to performance. However the strength of the response is greater when the reputation is lower. The reason for this is simple. When the manager’s reputation is very high the investor has already put most of their investable wealth into the managed portfolio. This means that the investor has only a little more wealth to move in response to an upward revision. And since assets under management are already high for managers with high reputations the percentage change in assets will be smaller. The estimate of \(b_2\) is significant and positive for low beginning reputations, meaning that the performance/flow relationship is convex. When reputations are high the relationship becomes concave.

Berk and Green (2004) claimed to find that a very high average level of initially perceived manager ability (over 600 basis points per year in expected excess return) is needed to match the performance/flow relationship documented in the literature. But this model delivers a convex relationship when
reputation is low, not when it is high.

5. Conclusion

In this paper I have presented a model in which the market is efficient and investors and managers are both equally informed and can trade in all the available assets. Since all assets are fairly priced (including the managed portfolios) the performance of the active managers using traditional performance measures will not be positive. Despite this investors still decide to hire active managers because they think it is possible that the managers have a better idea where the efficient frontier is than what investors would guess on their own. Investors update on this possibility based on the excess returns over a benchmark which, absent the manager, the investor would guess to be mean-variance efficient. The equilibrium contracts pay each manager a fixed-fraction of assets under management, just as we observe in practice. If investors are more risk averse than log then excess returns over this benchmark will lead to flows. The relationship between performance and flows will be convex if the typical manager has a relatively low reputation but the relationship can be concave for managers with very high reputations.
Appendix A. The Equilibrium

I will adopt the so-called martingale approach of Karatzas et al. (1987) and Cox and Huang (1989). The social planner’s problem is

\[
\max_{c, \phi} \lambda_I \mathbb{E}^I \int_0^\infty e^{-\rho t} u_I(c(t)) dt + \sum_{j=1}^M \lambda_j \mathbb{E}^j \int_0^\infty e^{-\rho t} u_j(\phi_j(t)) dt
\]

subject to the budget constraint

\[
\mathbb{E}^I \int_0^\infty H_I(t) \left( c(t) + \sum_{j=1}^M \phi_j(t) \right) dt \leq W(0).
\]

where \(W(0)\) is initial wealth and the superscripts on the expectation operators indicate whose probability beliefs are being used. I have expressed the budget constraint using the investors SDF and probability beliefs but that is arbitrary, we could have used the beliefs and SDF of any of the agents. For purposes of interpretation I will assume that all wealth belongs to the investor.

In order to solve the social planner’s problem we need to be able to compute all expectations using a single probability measure. Recall that state price densities can be described as

\[
H(t) = e^{-\int_0^t r(s) ds} \frac{dQ(t)}{dP(t)}
\]

where \(Q\) is the risk-neutral measure and \(P\) is the probability measures corresponding to the beliefs of some agent. Now we can see that the ratio of the state price density of the investor to that of either manager is also the likelihood ratio of the observed price data using the manager’s beliefs and the investor’s beliefs.

\[
\frac{H_I(t)}{H_j(t)} = \frac{dP_j(t)}{dP_I(t)} \quad \text{for } j = 1, \ldots, M \quad (A.1)
\]
These likelihood ratios can be used to change probability measures so that we can rewrite the social planner’s problem as

\[
\max_{c, \phi_j} \mathbb{E}^I \int_0^\infty \left\{ \lambda_I e^{-\rho t} u_I(c(t)) + \sum_{j=1}^M \lambda_j e^{-\rho t} u_j(\phi_j(t)) \frac{H_I(t)}{H_j(t)} \right\} dt
\]

subject to the budget constraint

\[
\mathbb{E}^I \int_0^\infty H_I(t) \left( c(t) + \sum_{j=1}^M \phi_j(t) \right) dt \leq W(0)
\]

from which the first-order conditions in the text are immediate.

**Appendix B. Bayesian Updating**

The proof below largely follows the proof found in Li (2007). Let \( p(0) \) be the prior probability that the manager is skilled and let \( p(t) \) be the posterior probability, after observing returns up to time \( t \). Bayes’ theorem says that

\[
p(t) = p(0) \frac{dP_M(t)}{p(0)dP_M(t) + (1-p(0))dP_B(t)}
\]

which I will rewrite as

\[
p(t) = \frac{\ell(t)}{\ell(t) + \frac{1-p(0)}{p(0)}}
\]

where \( \ell(t) \) is defined to be the likelihood ratio \( \frac{dP_M(t)}{dP_B(t)} \). Now note that the function \( g(x) = \frac{x}{x+k} \) has as it’s first and second derivatives

\[
g'(x) = \frac{g(x)(1-g(x))}{x} \quad \text{and} \quad g''(x) = -2\frac{g^2(x)(1-g(x))}{x^2}.
\]

With this in mind Ito’s formula gives

\[
\frac{dp(t)}{p(t)} = (1-p(t)) \left[ \frac{d\ell(t)}{\ell(t)} - p(t)\frac{d(\ell(t))}{\ell^2(t)} \right]. \quad \text{(B.1)}
\]
Now recall that $\ell(t) = \frac{dP_M(t)}{dP_B(t)} = \frac{H_B(t)}{H_M(t)}$. In terms of $dZ_I(t)$ we have that

$$\frac{dH_i(t)}{H_i(t)} = -r dt - \theta_i(t)^\top [\theta_I(t) - \theta_i(t)] dt + dZ_I(t) \quad i = M, B$$

The Ito formula gives us that

$$\frac{d\ell(t)}{\ell(t)} = (\theta_M(t) - \theta_B(t)) dZ_I(t) + (\theta_M(t) - \theta_B(t))^\top (\theta_I(t) - \theta_B(t)) dt$$

so that

$$\langle \frac{d\ell(t)}{\ell^2(t)} \rangle = (\theta_M(t) - \theta_B(t))^\top (\theta_M(t) - \theta_B(t)) dt$$

With this observation we can substitute in to (B.1) to obtain

$$\frac{dp(t)}{p(t)} = (1 - p(t))(\theta_M(t) - \theta_B(t)) dZ_I(t).$$

But now note that

$$\theta_M(t) - \theta_I(t) = (1 - p(t))(\theta_M(t) - \theta_B(t))$$

so that

$$\frac{dp(t)}{p(t)} = (\theta_M(t) - \theta_I(t))^\top dZ_I(t).$$

Comparing with (2) we see that $p(t)$ and $H_I(t)/H_M(t)$ have the same dynamics. Since $H_I(0) = H_M(0) = 1$ this gives us

$$\frac{p(t)}{p(0)} = \frac{H_I(t)}{H_M(t)}$$

as was desired.

**Appendix C. Power Utility**

Suppose both investor and manager have power utility with relative risk aversion coefficient $\gamma$. We can solve the first-order conditions stated in the text for the optimal consumption and fee.

$$c(t) = \left(e^{\rho t} yH_I(t)\right)^{-1/\gamma} \quad \text{(C.1)}$$
\[
\phi(t) = \left( e^{\frac{\gamma}{\eta} H_m(t)} \right)^{-1/\gamma}
\]

By the Ito formula we also have the following dynamics for \( c(t) \)
\[
\frac{dc(t)}{c(t)} = \left( \frac{r - \rho}{\gamma} + \frac{11}{2 \gamma} \left( \frac{1 + \gamma}{\gamma} \right) \theta_I(t) \theta_I(t)^\top \right) dt + \frac{1}{\gamma} \theta_I(t)^\top dZ_I(t).
\]

For simplicity in what follows we will write this as
\[
\frac{dc(t)}{c(t)} = \mu_c(t) dt + \frac{1}{\gamma} \theta_I(t)^\top dZ_I(t).
\]

Optimal wealth has two parts: the present value of investor consumption and the present value of manager fees. The present value of consumption is
\[
\frac{1}{H_I(t)} y^{-1/\gamma} \mathbb{E}_t^I \int_t^\infty e^{-\rho u/\gamma} H_I(u)^{(\gamma-1)/\gamma} du
\]

Define \( F(u, t) \) such that
\[
\mathbb{E}_t^I H_I(u)^{(\gamma-1)/\gamma} = H_I(t)^{(\gamma-1)/\gamma} F(u, t).
\]

and define \( G(t) \) as
\[
G(t) = \int_t^\infty e^{-\rho u/\gamma} F(t, u) du.
\]

Plugging this in above and using (C.1) we have that the present value of investor consumption is \( c(t)G(t) \).

Now write the dynamics of \( G(t) \) as
\[
\frac{dG(t)}{G(t)} = \mu_G(t) dt + \sigma_G(t) dZ_I(t)
\]

where the drive rate and volatility are, for the moment, unspecified. From the Ito formula we have
\[
\frac{d(c(t)G(t))}{c(t)G(t)} = \left[ \mu_G(t) + \mu_c(t) + \frac{1}{\gamma} \theta_I(t)^\top \sigma_G(t) \right] dt + \left[ \sigma_G(t) + \frac{1}{\gamma} \theta_I(t) \right]^\top dZ_I(t)
\]
But we also know that $c(t)G(t)$ is a wealth process which finances the consumptions stream $c(t)$. To prevent arbitrage then we must have the following.

$$\frac{d(c(t)G(t))}{c(t)G(t)} = \left(r - \frac{1}{G(t)}\right) dt + \left[\sigma_G(t) + \frac{1}{\gamma} \theta_I(t)\right] \theta_I(t) dt + dZ_I(t)$$ (C.2)

Equating the $dt$ portions of both these SDEs gives the following no-arbitrage condition on $G(t)$.

$$\mu_G(t) + \mu_c(t) + \frac{1 - \gamma}{\gamma} \theta_I(t)^\top \sigma_G(t) - \frac{1}{\gamma} \theta_I(t)^\top \theta_I(t) = r - \frac{1}{G(t)}$$ (C.3)

Using the specification for $x$ given in the text we can apply the Ito formula to $G(t)$ to obtain

$$\mu_G(t) = \frac{1}{2} G_{xx} x^2 (1 - x)^2 \sigma_x^\top \sigma_x + \frac{1}{2} G_{pp} p^2 (1 - p)^2 (2x - 1)^2 D^\top D$$

$$+ \frac{G_{xp}}{G} x (1 - x) p (2x - 1) \sigma_x^\top D$$

and

$$\sigma_G(t) = \frac{G_x}{G} x (1 - x) \sigma_x + \frac{G_p}{G} p (1 - p) (2x - 1) D.$$

Plugging these and the definition of $\mu_c(t)$ into (C.3) gives us the following.

$$\frac{1}{2} G_{xx} x^2 (1 - x)^2 \sigma_x^\top \sigma_x + \frac{1}{2} G_{pp} p^2 (1 - p)^2 (2x - 1)^2 D^\top D$$

$$+ G_{xp} x (1 - x) p (2x - 1) \sigma_x^\top D$$

$$+ G \left( \frac{r}{\gamma} - \frac{p}{\gamma} + \frac{1}{2} \frac{\gamma}{\gamma^2} \theta_I(t)^\top \theta_I(t) \right)$$

$$- G_x x (1 - x) \sigma_x^\top \theta_I(t) - G_p p (1 - p) (2x - 1) D^\top \theta_I(t) = -1$$

Now substituting in the definition of $\theta_I(t)$ and noting that

$$\theta_I(t)^\top \theta_I(t) = (\theta_B^\top \theta_B + 2p (2x - 1) \theta_B^\top D + p^2 (2x - 1)^2 D^\top D)$$

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gives us the PDE which must hold on $[0, 1] \times [0, 1]$.

But in order to solve this PDE numerically we need to know the behavior of the solution on the boundary of this region. At each of the four boundaries ($p = 1, p = 0, x = 1, \text{and } x = 0$) the PDE reduces to an ODE, each of which is straightforward to solve provided we know the solution at the corners.

At each corner all the derivatives disappear from the PDE and so we have the closed form solution

$$G = \frac{\gamma}{\rho - r(1 - \gamma) - \frac{1}{2}(\frac{1 - \gamma}{\gamma})\theta_I(t)\theta_I(t)}$$

where $\theta_I(t)^\top \theta_I(t)$ is equal to $\theta_B^\top \theta_B$ at both corners where $p = 0$, and when $p = 1$ we have $\theta_I(t)^\top \theta_I(t) = \theta_B^\top \theta_B + 2D^\top \theta_B + D^\top D$ when $x = 1$ and $\theta_I(t)^\top \theta_I(t) = \theta_B^\top \theta_B - 2D^\top \theta_B + D^\top D$ when $x = 0$.

Knowing all four corners means we can solve the equation on all four boundaries and then solve the PDE on the interior using standard techniques.

**Appendix D. Flows**

Recall from the text that

$$A(t) + \frac{1}{\gamma}c(t)G(p(t))p(t) + \frac{1}{\gamma}\phi(t)G(1).$$

We wish to obtain the approximate dynamics for $A(t)$. Since we know that the hedging demand is tiny we will ignore the terms involving derivatives of $G$. The present value of fee then has dynamics

$$\frac{d(\phi(t)G(t))}{\phi(t)G(t)} = \left(r - \frac{1}{G(t)}\right)dt + \frac{1}{\gamma}\theta_I(t)^\top [\theta_I(t)dt + dZ_I(t)]. \quad (D.1)$$
The term involving consumption is as follows.

\[
\frac{d(c(t)G(t)p(t))}{c(t)G(t)p(t)} = \frac{d(c(t)G(t))}{c(t)G(t)} + \frac{dp(t)}{p(t)} + \left\langle \frac{d(c(t)G(t))}{c(t)G(t)}, \frac{dp(t)}{p(t)} \right\rangle \\
= -\frac{dt}{G(p(t))} + rdt + \frac{1}{\gamma} \theta_t(t)^T [\theta_t(t)dt + dZ_I(t)] \\
+ (\theta_M - \theta_I(t))^T dZ_I(t) + \frac{1}{\gamma} \theta_I(t)^T [\theta_M - \theta_I(t)]dt
\]

which, after some cancellation, yields

\[
d(c(t)G(t)p(t)) = -c(t)p(t)dt + c(t)G(t)p(t)dR(t) + \frac{\gamma - 1}{\gamma} c(t)G(t)dp(t).
\]

Combining this with (D.1) gives the following expression for the dynamics of \(A(t)\).

\[
da(t) = -\frac{c(t)p(t) + \phi(t)}{\gamma} dt + A(t) \left[ \frac{\gamma - 1}{\gamma} rdt + \frac{1}{\gamma} dR_M(t) \right] + \frac{\gamma - 1}{\gamma^2} c(t)G(p(t))dp(t).
\]

The SDE in the text is obtained by substituting in for \(dp(t)\).
References


